Characterizing Mueller matrices in Polarimetry

Ghislain R. Franssens

Belgian Institute for Space Aeronomy
Ringlaan 3, 1180 Brussels, Belgium

COST Action MP1104 - WG4 Meeting, 19-21/08/2013

Subject

A polarimetric measurement of a medium or surface results in a real 4×4 matrix, called the *Stokes scattering matrix* or the *Mueller matrix* of the object.

Talk: Introduction to the mathematical structure of Mueller matrices.

Subject

A polarimetric measurement of a medium or surface results in a real 4×4 matrix, called the *Stokes scattering matrix* or the *Mueller matrix* of the object.
Talk: Introduction to the mathematical structure of Mueller matrices.

- **Setting**: Theory of Transversal Polarization of partially coherent plane waves.
- **Topic**: Modeling of the polarization altering properties of *linear* media and surfaces.
- **Model**: Using real 4×4 Mueller matrices.

Transversal Polarization Formalisms

Jones formalism:
Light:
(i) Totally polarized plane waves
(ii) Represented by a complex 2×1 *Jones vector*
Medium:
(i) Linear and "non-depolarizing" ($p_{out} = 1 = p_{in}$)
(ii) Represented by a complex 2×2 *Jones matrix*

Stokes/Mueller formalism:
Light:
(i) Partially polarized plane waves
(ii) Represented by a real 4×1 *Stokes vector*
Medium:
(i) Linear
(ii) Represented by a real 4×4 *Mueller matrix*

Stokes Vectors

Definition

A Stokes vector $S = [I, Q, U, V]^\top$ is a real 4×1 vector satisfying: (i) $I \geq 0$ and (ii) $I^2 - \left(Q^2 + U^2 + V^2\right) \geq 0$ (or $p \leq 1$).

We denote the set of Stokes vectors by $\mathcal{S}$.

Convenient representation of a Stokes vector S:

$$S = I \begin{bmatrix} 1 \\ p\mathbf{u} \end{bmatrix},$$

with *intensity* $I \geq 0$, *degree of polarization* $0 \leq p \leq 1$ and *polarization state* $\mathbf{u} \in S^2$ (Poincaré sphere).

Stokes Vectors

Definition

A Stokes vector $S = [I, Q, U, V]^\top$ is a real 4×1 vector satisfying: (i) $I \geq 0$ and (ii) $I^2 - \left(Q^2 + U^2 + V^2\right) \geq 0$ (or $p \leq 1$).

We denote the set of Stokes vectors by $\mathcal{S}$.

Convenient representation of a Stokes vector S:

$$S = I \begin{bmatrix} 1 \\ p\mathbf{u} \end{bmatrix},$$

with *intensity* $I \geq 0$, *degree of polarization* $0 \leq p \leq 1$ and *polarization state* $\mathbf{u} \in S^2$ (Poincaré sphere).

Stokes vectors in the Theory of Transversal Polarization (TTP) correspond with *four-momentum vectors* in the Special Theory of Relativity (STR). Hence, TTP and STR share the *same mathematics*!

TTP and STR Correspondence

Quantity	TTP	STR
I	Intensity	Rel. energy E divided by c
$I_p \triangleq pI$	Polarization intensity	Rel. momentum $\lVert \mathbf{p} \rVert$
p	Degree of polarization	Normalized speed $\lVert \mathbf{v} \rVert / c$
$\mathbf{u}$	Polarization state	Unit velocity vector $\mathbf{v} / \lVert \mathbf{v} \rVert$
$\beta \triangleq \operatorname{artanh} p$	Lorentzian angle of pol.	Rapidity β
$\gamma \triangleq \frac{1}{\sqrt{1-p^2}}$	Lorentzian factor of pol.	Time dilatation factor γ
$\lVert S \rVert_{1,3}$	Lorentzian length of S	Rest energy E_0 divided by c

Table : Correspondence between a **Stokes vector** $S = I\,[1, p\mathbf{u}]^\top$ in the Theory of Transversal Polarization (TTP) and the **four-momentum vector** $P = [E/c, \mathbf{p}]^\top$ in the Special Theory of Relativity (STR), of a *uniformly moving particle* with rest mass $m_0 = I/(\gamma c)$, relativistic mass $m = \gamma m_0 = I/c$, velocity vector $\mathbf{v} = (pc)\,\mathbf{u}$ and relativistic momentum vector $\mathbf{p} = m\mathbf{v} = I_p\mathbf{u}$.

Definition

A *Mueller matrix* is a real 4×4 matrix that transforms any Stokes vector into a Stokes vector. Denote the set of Mueller matrices by $\mathcal{M}$.

Definition

A *Mueller matrix* is a real 4×4 matrix that transforms any Stokes vector into a Stokes vector. Denote the set of Mueller matrices by $\mathcal{M}$.

Properties

- The set $\mathcal{M}$, together with matrix multiplication, is a *monoid*.
- Non-singular Mueller matrices represent Helmholtz-reciprocal media and form a *(Lie) group*.
- The *orthochronous Lorentz group* $O_+(1,3)$ is a subgroup of the group of Mueller matrices.
- The group of Mueller matrices is (much) *larger* than $O_+(1,3)$.
- An analytical characterization for $\mathcal{M}$ *has not been given yet*.

Theorem

[van der Mee, 1993] Let $M \in M(4, \mathbb{R})$ *satisfying* $m_{11}^2 \geq m_{12}^2 + m_{13}^2 + m_{14}^2$, $G \triangleq diag[1, -1, -1, -1]$ *and* $A \triangleq GM^\top GM$. *Then* $M \in \mathcal{M}$ *iff one of the following two situations occurs:*

(i) A has one real eigenvalue λ_0*, corresponding to a positive eigenvector, and three real eigenvalues* λ_1*,* λ_2*,* λ_3*, corresponding to negative eigenvectors, and* $\lambda_0 \geq \max(0, \lambda_1, \lambda_2, \lambda_3)$.

(ii) A has four real eigenvalues λ*,* λ*,* μ *and* ν *but is not diagonalizable. The eigenvectors corresponding to* μ *and* ν *are negative and to the double eigenvalue* λ *corresponds a Jordan block of size* 2 *with positive sign. Moreover,* $\lambda \geq \max(0, \mu, \nu)$.

Mueller Matrices

Motivation for an analytical characterization

- Need for a simpler test than van der Mee's result, directly in terms of the Mueller matrix elements itself (for reasons of error propagation through the test algorithm).
- Interpretation in terms of simple polarization effects (e.g., via factoring).
- Understanding the mathematical structure of Mueller matrices (e.g., the Lie group structure of the non-singular matrices).

Mueller Matrices

Which are equivalent to a Jones matrix

Mueller matrices which have a corresponding Jones matrix are called *Jones-Mueller matrices*.

Mueller Matrices

Which are equivalent to a Jones matrix

Mueller matrices which have a corresponding Jones matrix are called *Jones-Mueller matrices*.

Non-singular Jones-Mueller matrices have the form aL, with $a > 0$ and $L \in SO_+(1,3)$. Explicitly,

$$M = a\gamma \begin{bmatrix} 1 & p\mathbf{x}^\top \\ p\mathbf{y} & \gamma^{-1}R + \left(1 - \gamma^{-1}\right)\mathbf{y}\mathbf{x}^\top \end{bmatrix},$$

with $\mathbf{x}, \mathbf{y}$ Euclidean unit vectors, $0 \leq p < 1$, $1 \leq \gamma \triangleq 1/\sqrt{1-p^2} < +\infty$, $R \in SO(3)$ and $\mathbf{y} = R\mathbf{x}$.

Mueller Matrices

Which are equivalent to a Jones matrix

Mueller matrices which have a corresponding Jones matrix are called *Jones-Mueller matrices*.

Non-singular Jones-Mueller matrices have the form aL, with $a > 0$ and $L \in SO_+(1,3)$. Explicitly,

$$M = a\gamma \left[\begin{array}{cc} 1 & p\mathbf{x}^\top \\ p\mathbf{y} & \gamma^{-1}R + \left(1 - \gamma^{-1}\right)\mathbf{y}\mathbf{x}^\top \end{array} \right],$$

with $\mathbf{x}, \mathbf{y}$ Euclidean unit vectors, $0 \leq p < 1$,
$1 \leq \gamma \triangleq 1/\sqrt{1-p^2} < +\infty$, $R \in SO(3)$ and $\mathbf{y} = R\mathbf{x}$.

The subgroup corresponding with $a = 1$ and $p = 0$ is $SO(3)$ and represents *retarders* (birefringence).
The subset corresponding with $a = 1$ and $R = I_3$ are the Lorentz boost matrices, which represent *diattenuators* (dichroism).

Define $\|M_3\|_{op} \triangleq \max_{\forall \mathbf{u}\in\mathcal{S}^2} \|M_3\mathbf{u}\|_{3,0}$.

Theorem

Let $\mathbf{x}, \mathbf{y} \in \mathcal{S}^2$ *and* $M_3 \in M(3;\mathbb{R})$. *If* $M \in M(4;\mathbb{R})$ *is of the form*

$$M = a \begin{bmatrix} 1 & b\mathbf{x}^\top \\ c\mathbf{y} & M_3 + bc\mathbf{y}\mathbf{x}^\top \end{bmatrix},$$

with

$$0 \le a, \;\; 0 \le b \le 1, \;\; 0 \le c \le 1,$$

$$\|M_3\|_{op} \le (1-b)(1-c),$$

then M is a Mueller matrix.

Analytic Characterizations

Some Necessary conditions satisfied by Mueller matrices

Infinitely many necessary conditions can be derived for the elements of a Mueller matrix by substituting particular values for p_{in} and u_{in} in the conditions $0 \leq I_{out}$ and $p_{out} \leq 1$.
The following is a particular, but useful, result.

Theorem

Let $\mathbf{x}, \mathbf{y} \in \mathcal{S}^2$ and $M_3 \in M(3;\mathbb{R})$. Any $M \in \mathcal{M}$ is necessarily of the form

$$M = a \begin{bmatrix} 1 & b\mathbf{x}^\top \\ c\mathbf{y} & M_3 + bc\mathbf{y}\mathbf{x}^\top \end{bmatrix},$$

with

$$0 \leq a, \;\; 0 \leq b \leq 1, \;\; 0 \leq c \leq 1.$$

If $b = 1$ or $c = 1$, then $M_3 = 0$.

Analytic Characterizations

Sufficient condition for M to be NOT a Mueller matrix

Define $\|M_3\|_{\min} \triangleq \min_{\forall \mathbf{u} \in \mathcal{S}^2} \|M_3 \mathbf{u}\|_{3,0}$.

Theorem

Let $\mathbf{x}, \mathbf{y} \in \mathcal{S}^2$ *and* $M_3 \in M(3; \mathbb{R})$. *If* $M \in M(4; \mathbb{R})$ *is of the form*

$$M = a \begin{bmatrix} 1 & b\mathbf{x}^\top \\ c\mathbf{y} & M_3 + bc\mathbf{y}\mathbf{x}^\top \end{bmatrix},$$

with

$$0 \le a, \quad 0 \le b \le 1, \quad 0 \le c \le 1,$$

$$\|M_3\|_{\min} > (1+b)(1+c),$$

then M is NOT a Mueller matrix.

Analytic Characterizations

A Necessary and Sufficient condition for M to be a Mueller matrix

Let $\mathcal{S}_1 \subset \mathcal{S}$ denote the set of Stokes vectors having degree of polarization 1.

Theorem

In order that a $M \in M(4;\mathbb{R})$ *is in* $\mathcal{M}$*, it is necessary and sufficient that* M *maps* $\mathcal{S}_1 \to \mathcal{S}$*.*

Analytic Characterizations

Necessary and Sufficient condition for a subset of Mueller matrix

Theorem

Let $\mathbf{x} \in \mathcal{S}^2$*. For a* $M \in M(4;\mathbb{R})$ *of the form*

$$M = a \begin{bmatrix} 1 & b\mathbf{x}^\top \\ c\mathbf{x} & dI_3 + bc\mathbf{x}\mathbf{x}^\top \end{bmatrix}$$

to be a Mueller matrix, it is necessary and sufficient that

$$0 \le a, \;\; 0 \le b \le 1, \;\; 0 \le c \le 1$$

and

$$\max\left(\left| c - \frac{d}{1-b} \right|, \left| c + \frac{d}{1+b} \right| \right) \le 1.$$

Analytic Characterizations

Necessary and Sufficient condition: principle

- Let $\mathbf{u}_{in} \in \mathcal{S}^2$ be an input polarization state where the output degree of polarization p_{out} takes on its maximal value $(p_{out})_{\max}$.
- The *sufficient* condition is then equivalent to $(p_{out})_{\max} \leq 1$.
- That the condition $(p_{out})_{\max} \leq 1$ is also *necessary* is implied by the existence of the input polarization state $\mathbf{u}_{in}$ where p_{out} reaches $(p_{out})_{\max}$.

Analytic Characterizations

Necessary and Sufficient condition: principle

- Let $\mathbf{u}_{in} \in \mathcal{S}^2$ be an input polarization state where the output degree of polarization p_{out} takes on its maximal value $(p_{out})_{\max}$.
- The *sufficient* condition is then equivalent to $(p_{out})_{\max} \leq 1$.
- That the condition $(p_{out})_{\max} \leq 1$ is also *necessary* is implied by the existence of the input polarization state $\mathbf{u}_{in}$ where p_{out} reaches $(p_{out})_{\max}$.

When we search for the necessary and sufficient condition for a general Mueller matrix, we run into very complicated and totally unpractical expressions.
A more clever approach is needed to obtain the necessary and sufficient condition for a general Mueller matrix.

Analytic Characterizations

Conclusions so far

- We have an optimal sufficient condition for a matrix to be a Mueller matrix.
- We have an optimal sufficient condition for a matrix NOT to be a Mueller matrix.
- We know that the conditions $0 \leq a,\ 0 \leq b \leq 1, 0 \leq c \leq 1$ are both necessary and sufficient.
- We have the necessary and sufficient condition for a particular subset of simple Mueller matrices.
- We have the analytical characterization of the special set of Jones-Mueller matrices.

Vectorial Radiative Transfer Theory

- Combining the transversal polarization of partially coherent plane waves, in terms of the Stokes/Mueller formalism, with the phenomenological theory of (stationary) scalar radiative transfer goes back to [Chandrasekhar 1950] and [Rozenberg 1955]. The result is the well-known Vectorial Radiative Transfer (VRT) eq.

Vectorial Radiative Transfer Theory

- Combining the transversal polarization of partially coherent plane waves, in terms of the Stokes/Mueller formalism, with the phenomenological theory of (stationary) scalar radiative transfer goes back to [Chandrasekhar 1950] and [Rozenberg 1955]. The result is the well-known Vectorial Radiative Transfer (VRT) eq.
- This model however possesses certain mathematical subtleties, which are not mentioned in the standard text books and have not been recognized so far.

Vectorial Radiative Transfer Theory

- Combining the transversal polarization of partially coherent plane waves, in terms of the Stokes/Mueller formalism, with the phenomenological theory of (stationary) scalar radiative transfer goes back to [Chandrasekhar 1950] and [Rozenberg 1955]. The result is the well-known Vectorial Radiative Transfer (VRT) eq.
- This model however possesses certain mathematical subtleties, which are not mentioned in the standard text books and have not been recognized so far.
- These subtleties are related to the global topology of the manifold of the underlying Lie group of Mueller matrices.

Vectorial Radiative Transfer Theory

- Combining the transversal polarization of partially coherent plane waves, in terms of the Stokes/Mueller formalism, with the phenomenological theory of (stationary) scalar radiative transfer goes back to [Chandrasekhar 1950] and [Rozenberg 1955]. The result is the well-known Vectorial Radiative Transfer (VRT) eq.
- This model however possesses certain mathematical subtleties, which are not mentioned in the standard text books and have not been recognized so far.
- These subtleties are related to the global topology of the manifold of the underlying Lie group of Mueller matrices.
- As a consequence, a possible discrepancy can arise between (i) the solution of the VRT equation and (ii) an in situ measurement of the Stokes vector in the medium.

Vectorial Radiative Transfer Theory

- Combining the transversal polarization of partially coherent plane waves, in terms of the Stokes/Mueller formalism, with the phenomenological theory of (stationary) scalar radiative transfer goes back to [Chandrasekhar 1950] and [Rozenberg 1955]. The result is the well-known Vectorial Radiative Transfer (VRT) eq.
- This model however possesses certain mathematical subtleties, which are not mentioned in the standard text books and have not been recognized so far.
- These subtleties are related to the global topology of the manifold of the underlying Lie group of Mueller matrices.
- As a consequence, a possible discrepancy can arise between (i) the solution of the VRT equation and (ii) an in situ measurement of the Stokes vector in the medium.
- This is another motivation for studying the set of Mueller matrices on a deeper mathematical level and to search for an analytic characterization of these matrices.

Vectorial Lambert-Beer (VLB)

A. Infinitesimal model

In a medium without scattering and emission, the VRT eq. reduces to (along a given fixed LOS)

$$\frac{d}{dz}S\left(z\right) = -K\left(z\right)S\left(z\right). \tag{1}$$

Eq. (1) describes the transport through our medium over an infinitesimal extent. Eq. (1) is an *infinitesimal model*.

Vectorial Lambert-Beer (VLB)

A. Infinitesimal model

In a medium without scattering and emission, the VRT eq. reduces to (along a given fixed LOS)

$$\frac{d}{dz} S(z) = -K(z) S(z) . \tag{1}$$

Eq. (1) describes the transport through our medium over an infinitesimal extent. Eq. (1) is an *infinitesimal model*.

B. Finite model

From an experimental point of view, there must exist a Mueller matrix M such that

$$S(z) = M(z, z_0) S(z_0) , \tag{2}$$

relating the Stokes vectors at z_0 and z.

Eq. (2) describes the transport through our medium over a finite extent. Eq. (2) is (part of) the *finite model*.

A Fundamental Question

Question:

Is the VRT infinitesimal model equivalent to the VRT finite model ?

Answer:

No (in general).

How Can It Go Wrong ?

In the example of the VLB law, with constant extinction matrix K, the solution of its infinitesimal model is

$$S(z) = \exp\left(-K\,(z - z_0)\right) S(z_0)\,.$$

How Can It Go Wrong ?

In the example of the VLB law, with constant extinction matrix K, the solution of its infinitesimal model is

$$S(z) = \exp\left(-K\,(z - z_0)\right) S(z_0)\,.$$

Although this is the correct solution to the infinitesimal model, this solution could potentially differ from the finite model.

How Can It Go Wrong ?

In the example of the VLB law, with constant extinction matrix K, the solution of its infinitesimal model is

$$S(z) = \exp\left(-K\,(z - z_0)\right) S(z_0)\,.$$

Although this is the correct solution to the infinitesimal model, this solution could potentially differ from the finite model.

This would happen for Mueller matrices M that cannot be reached by the exponential function.
So, if such a medium is characterized by an unreachable Mueller matrix, then any solution method (numerically or analytically) will produce the wrong answer $\Rightarrow$ disagreement at experimental validation!
There is no apparent reason why the Mueller matrix of such a medium should be in the range of the matrix exponential function.

The End

THANK YOU

References

[1] Barakat R., Opt. Commun., 38, 3, 159–161, 1981
[2] Lu S-Y. and R.A. Chipman, J. Opt. Soc. Am. A, 13, , 1106–1113, 1996
[3] Givens C.R. and A.B. Kostinski, J. Mod. Opt., 41, 3, 471–481, 1993
[4] van der Mee C.V.M., J. Math. Phys., 34, 11, 5072–5088, 1993
[5] Degl'Innocenti E.L. and J.C. del Toro Iniesta, J. Opt. Soc. Am. A, 15, 2, 533–537, 1998
[6] Yushtin K.E. and S.N. Savenkov, Proc. Int. Conf. on Mathematical Methods in Electromagnetic Theory, MMET 98, , 435–437, 1998
[7] Twietmeyer K.M. and R.A. Chipman, Opt. Express, 16, 15, 11589–11603, 2008
[8] Boulvert F. and G. Le Brun and B. Le Jeune and J. Cariou and L. Martin, Opt. Commun., 282, 5, 692–704, 2009
[9] Anderson D.G.M. and R. Barakat, J. Opt. Soc. Am. A, 11, , 2305–2319, 1994
[10] Simon R., Opt. Commun., 42, 5, 293–297, 1982
[11] Franssens, G.R., Int. J. Remote Sensing (submitted)

Lie Group Concepts

- Is: (i) algebraically a *group*, (ii) geometrically a (smooth) *manifold*.

Lie Group Concepts

- Is: (i) algebraically a *group*, (ii) geometrically a (smooth) *manifold*.
- Each point of the manifold $\Leftrightarrow$ an element of the *Lie group*.

Lie Group Concepts

- Is: (i) algebraically a *group*, (ii) geometrically a (smooth) *manifold*.
- Each point of the manifold $\Leftrightarrow$ an element of the *Lie group*.
- Tangent plane to the manifold at the identity element is a local linearization of the manifold. Its points are given the structure of an algebra: the *Lie algebra* of the Lie group.

Lie Group Concepts

- Is: (i) algebraically a *group*, (ii) geometrically a (smooth) *manifold*.
- Each point of the manifold $\Leftrightarrow$ an element of the *Lie group*.
- Tangent plane to the manifold at the identity element is a local linearization of the manifold. Its points are given the structure of an algebra: the *Lie algebra* of the Lie group.
- Each point of this tangent plane $\Leftrightarrow$ an element of the *Lie algebra*.

Lie Group Concepts

- Is: (i) algebraically a *group*, (ii) geometrically a (smooth) *manifold*.
- Each point of the manifold $\Leftrightarrow$ an element of the *Lie group*.
- Tangent plane to the manifold at the identity element is a local linearization of the manifold. Its points are given the structure of an algebra: the *Lie algebra* of the Lie group.
- Each point of this tangent plane $\Leftrightarrow$ an element of the *Lie algebra*.
- Locally, the group structure of a Lie group G is reflected in the structure of its Lie algebra $\mathfrak{g}$.

Lie Group Concepts

- Is: (i) algebraically a *group*, (ii) geometrically a (smooth) *manifold*.
- Each point of the manifold $\Leftrightarrow$ an element of the *Lie group*.
- Tangent plane to the manifold at the identity element is a local linearization of the manifold. Its points are given the structure of an algebra: the *Lie algebra* of the Lie group.
- Each point of this tangent plane $\Leftrightarrow$ an element of the *Lie algebra*.
- Locally, the group structure of a Lie group G is reflected in the structure of its Lie algebra $\mathfrak{g}$.
- *Exponentiation* of an element in $\mathfrak{g}$ produces an element of G.

Lie Group Concepts

- Is: (i) algebraically a *group*, (ii) geometrically a (smooth) *manifold*.
- Each point of the manifold $\Leftrightarrow$ an element of the *Lie group*.
- Tangent plane to the manifold at the identity element is a local linearization of the manifold. Its points are given the structure of an algebra: the *Lie algebra* of the Lie group.
- Each point of this tangent plane $\Leftrightarrow$ an element of the *Lie algebra*.
- Locally, the group structure of a Lie group G is reflected in the structure of its Lie algebra $\mathfrak{g}$.
- *Exponentiation* of an element in $\mathfrak{g}$ produces an element of G.
- Problem: the map $exp : \mathfrak{g} \rightarrow G$ is often *not surjective* (i.e. "some matrices M cannot be reached").

Lie Group Concepts

- Is: (i) algebraically a *group*, (ii) geometrically a (smooth) *manifold*.
- Each point of the manifold $\Leftrightarrow$ an element of the *Lie group*.
- Tangent plane to the manifold at the identity element is a local linearization of the manifold. Its points are given the structure of an algebra: the *Lie algebra* of the Lie group.
- Each point of this tangent plane $\Leftrightarrow$ an element of the *Lie algebra*.
- Locally, the group structure of a Lie group G is reflected in the structure of its Lie algebra $\mathfrak{g}$.
- *Exponentiation* of an element in $\mathfrak{g}$ produces an element of G.
- Problem: the map $exp : \mathfrak{g} \rightarrow G$ is often *not surjective* (i.e. "some matrices M cannot be reached").
- Is a consequence of the global *topology* of the manifold.

Lie Group Concepts

- Is: (i) algebraically a *group*, (ii) geometrically a (smooth) *manifold*.
- Each point of the manifold $\Leftrightarrow$ an element of the *Lie group*.
- Tangent plane to the manifold at the identity element is a local linearization of the manifold. Its points are given the structure of an algebra: the *Lie algebra* of the Lie group.
- Each point of this tangent plane $\Leftrightarrow$ an element of the *Lie algebra*.
- Locally, the group structure of a Lie group G is reflected in the structure of its Lie algebra $\mathfrak{g}$.
- *Exponentiation* of an element in $\mathfrak{g}$ produces an element of G.
- Problem: the map $exp : \mathfrak{g} \to G$ is often *not surjective* (i.e. "some matrices M cannot be reached").
- Is a consequence of the global *topology* of the manifold.
- Key concepts: (i) *compactness*, (ii) *connectedness* and (iii) *simply connectedness* of the manifold.

Lie Group Topology Effects

- If a Lie group manifold is *not connected*, then *exp* cannot reach group elements on the non-identity component.

Example: full Lorentz group $O(1,3)$: 4 components. Its subgroup $O_+(1,3)$: 2 components.

Lie Group Topology Effects

- If a Lie group manifold is *not connected*, then *exp* cannot reach group elements on the non-identity component.

Example: full Lorentz group $O(1,3)$: 4 components. Its subgroup $O_+(1,3)$: 2 components.

- If a component of a Lie group is also *not simply connected*, then *exp* is not guaranteed (by general Lie theory) to be surjective.

Example: each component of the Lorentz group is not simply connected (since $SO(3)$ is not simply connected).

Lie Group Topology Effects

- If a Lie group manifold is *not connected*, then *exp* cannot reach group elements on the non-identity component.

Example: full Lorentz group $O(1,3)$: 4 components. Its subgroup $O_+(1,3)$: 2 components.

- If a component of a Lie group is also *not simply connected*, then *exp* is not guaranteed (by general Lie theory) to be surjective.

Example: each component of the Lorentz group is not simply connected (since $SO(3)$ is not simply connected).

- The map *exp* may exceptionally be surjective.

Example: the identity component $SO_+(1,3)$ of the Lorentz group.

Summary of the VRT problem

The illness:

- The VRT equation is an infinitesimal model and as such, it is a local model.
- If the group underlying an equation has trivial topology, then: infinitesimal model $\Leftrightarrow$ finite model.
- The group of Mueller matrices underlying the VRT problem is not fully known, but it is already known that it has non-trivial topology (non-compact, not connected and not simply connected).
- Consequently, infinitesimal VRT model $\not\Leftrightarrow$ finite model!

Summary of the VRT problem

The illness:

- The VRT equation is an infinitesimal model and as such, it is a local model.
- If the group underlying an equation has trivial topology, then: infinitesimal model $\Leftrightarrow$ finite model.
- The group of Mueller matrices underlying the VRT problem is not fully known, but it is already known that it has non-trivial topology (non-compact, not connected and not simply connected).
- Consequently, infinitesimal VRT model $\not\Leftrightarrow$ finite model!

The cure:

- Supply the information that got stripped away when formulating the infinitesimal model.
- The lost information is: the global structure of the manifold of Mueller matrices.
- Determine the component on which the Mueller matrices of the medium are located (i.e., choose the right "neighborhood").
- Reformulate the VRT equation on the tangent plane at *an element* of this component and solve as usual!

What is a monoid and a group ?

A **monoid** is:

- a non-empty *set* S

A **group** is:

What is a monoid and a group ?

A **monoid** is:

- a non-empty *set* S
- with a *binary operation* $\times : S \times S \to S$

A **group** is:

What is a monoid and a group ?

A **monoid** is:

- a non-empty *set* S
- with a *binary operation* $\times : S \times S \to S$
- in which $\times$ is *associative*

A **group** is:

What is a monoid and a group ?

A **monoid** is:

- a non-empty *set* S
- with a *binary operation* $\times : S \times S \to S$
- in which $\times$ is *associative*
- and a *unity* element $1 \in S$ such that $g \times 1 = g = 1 \times g, \forall g \in S$,

A **group** is:

What is a monoid and a group ?

A **monoid** is:

- a non-empty *set* S
- with a *binary operation* $\times : S \times S \to S$
- in which $\times$ is *associative*
- and a *unity* element $1 \in S$ such that $g \times 1 = g = 1 \times g, \forall g \in S$,

A **group** is:

- a monoid S

What is a monoid and a group ?

A **monoid** is:

- a non-empty *set* S
- with a *binary operation* $\times : S \times S \to S$
- in which $\times$ is *associative*
- and a *unity* element $1 \in S$ such that $g \times 1 = g = 1 \times g, \forall g \in S$,

A **group** is:

- a monoid S
- and $\forall g \in S$ exists an *inverse* element $g^{-1} \in S$ such that $g \times g^{-1} = 1 = g^{-1} \times g$.